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APPLICATIONS OF THE RELAXATION TECHNIQUE IN FLUID MECHANICS

by

John S. McNown[1], En-Yun Hsu[2] and Chia-Shun Yih[3]

Introduction

Relaxation, a numerical method of integration, can be used to obtain solutions for a wide variety of problems which are not solvable by standard methods. In employing this method, a network of values is assumed for the function sought, and is then systematically corrected as the errors are "relaxed". Based on the calculus of finite differences, this method was brought to fruition in England by Southwell and his associates [1, 2]; it has been used in solving a variety of problems in elasticity, heat transfer, and fluid mechanics during recent years. Although only particular solutions can be obtained by such methods, and although the computations are time consuming, relaxation methods are effective in cases for which a general solution cannot be obtained by direct methods.

Partial differential equations such as the Laplacian, the Poisson, and the biharmonic can be integrated using the relaxation procedure. These equations are often encountered in engineering, and numerous general solutions have been obtained for simple boundary conditions. However, general solutions are usually unobtainable if the boundary conditions are complex. It is in such cases that numerical solutions are particularly valuable because in their application a complex boundary condition can be fulfilled almost as easily as a simple one. In this discussion is presented only the method of solving the Laplace equation, the most frequently encountered in the analysis of fluid flow. Although this is the simplest of the several equations, the manner of solving it is nonetheless indicative of the general method. Applications of this differential equation are restricted to potential flows such as seepage, flow through boundary contractions, efflux through slots or orifices, certain types of wave motion, and flow over

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weirs. Only for two-dimensional and axisymmetric flow patterns is it feasible to obtain solutions.

Resolution of any significant problem by means of the relaxation process will require at least many hours and perhaps as long as several months, yet the individual steps in the process are few and easily mastered. These steps are presented in some detail, herein, from the initial assumed network of values, through the elimination of residual errors and the advances to successively finer nets, to the eventual satisfaction of both the differential equation and the assigned boundary conditions. A number of applications are discussed in the last section, the analyses having been conducted at the Iowa Institute of Hydraulic Research by graduate students and members of the staff.

Much of the work described herein was supported by the Office of Naval Research under contract N8onr-500. Among those who assisted in the time-consuming computations are A.H. Abul-Fetouh, A. LeClerc, Y. C. Soong, W. Hsieh, S. Ince, S. C. Liu, and T. Sarpkaya, either now or formerly graduate students and research assistants at the State University of Iowa. Dr. Hunter Rouse, Director of the Institute, suggested or directed several of the studies.

Essential Elements of Finite Difference Theory

The goal in applying the relaxation method is the attainment of a network of values of a function which satisfies a given partial differential equation at each point of a dis-

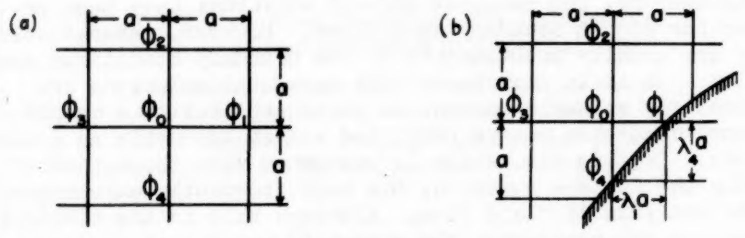

Fig. 1. Pattern of function values;
(a) regular, and (b) irregular

crete network covering the region, and which also satisfies a given set of boundary conditions. The number of points must be large enough (or the mesh fine enough) that the assumption of a linear variation of the function between adjacent points is justifiable. For such a network, the theory of finite differences can be successfully used.

An algebraic expression for the Laplacian of a particular function, say φ, can be obtained quite simply. With reference to Fig. 1a, the approximate value of

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \tag{1}$$

is sought for the central point in terms of the several values of φ in the immediate vicinity, the mesh being square and the lines spaced a distance a apart. If the coordinates of the central point are designated as x_0 and y_0, at the points $(x_0 + a/2, y_0)$ and $(x_0 - a/2, y_0)$

$$\frac{\partial \varphi}{\partial x} \approx \frac{\varphi_1 - \varphi_0}{a} \quad , \quad \frac{\partial \varphi}{\partial x} \approx \frac{\varphi_0 - \varphi_3}{a}$$

respectively. The second partial derivative of φ can now be obtained in a similar manner:

$$\frac{\partial^2 \varphi}{\partial x^2} \approx \frac{\frac{\varphi_1 - \varphi_0}{a} - \frac{\varphi_0 - \varphi_3}{a}}{a} = \frac{\varphi_1 + \varphi_3 - 2\varphi_0}{a^2}$$

In the same way, one obtains an expression for the second partial with respect to y:

$$\frac{\partial^2 \varphi}{\partial y^2} \approx \frac{\varphi_2 + \varphi_4 - 2\varphi_0}{a^2}$$

Finally, the equivalent form for Eq. (1) expressed in finite-difference form is:

$$a^2 \nabla^2 \varphi^2 \approx \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 \tag{2}$$

Although Eq. (2) is an approximation, equality is closely approached as the spacing is reduced. For a given set of φ-values, either known or assumed, $a^2 \nabla^2 \varphi$ can be evaluated by means of Eq. (2), and its difference from zero (or other assigned value for the Poisson equation) noted. The successive elimination of these undesired residues is discussed in the following section.

Corresponding relationships can also be derived for axisymmetric flow, but an additional complication is involved. The appropriate form for the differential equation of the velocity potential is as follows:

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial x^2} = 0 \tag{3}$$

in which x is measured parallel to the axis of symmetry and r at right angles to it. Although both the velocity potential

and the stream function Ψ for two-dimensional flow satisfy the same equation, the Stokes stream function, for axisymmetric flow, satisfies an equation similar to Eq. (3), but differing in the sign of the first-order term:

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{4}$$

At the point (x_0 , r_0) , as for the two-dimensional case (Fig. 1a),

$$\frac{\partial \varphi}{\partial r} \approx \frac{\varphi_2 - \varphi_4}{2a} \qquad \frac{\partial \varphi}{\partial x} \approx \frac{\varphi_1 - \varphi_3}{2a}$$

If this value is substituted into Eq. (3), together with the finite-difference equivalents of the second partial derivatives,

$$a^2 \nabla^2 \varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 + \frac{1}{2n}(\varphi_2 - \varphi_4) = 0 \tag{5}$$

the corresponding form for Ψ is

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0 - \frac{1}{2n}(\psi_2 - \psi_4) = 0 \tag{6}$$

in which n is r_0/a - the number of subdivisions between the axis of symmetry and the point (x_0 , r_0) .

If the four neighboring points are not equidistant from a given point, as for a point near an irregular boundary, the forms of Eqs. (2), (5), and (6) must be modified. As shown in Fig. 1b, the lengths of two of the legs on the star may be less than a . If these lengths are designated as $\lambda_1 a$ and $\lambda_4 a$ $(0<\lambda<1)$, the appropriate form for the Laplacian operator for two-dimensional flow is

$$a^2\nabla^2\varphi \approx \frac{\varphi_1}{\lambda_1} + \varphi_2 + \varphi_3 + \frac{\varphi_4}{\lambda_4} - \varphi_0\left(2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_4}\right) \tag{7}$$

and for axisymmetric flow,

$$a^2\nabla^2\varphi \approx \frac{\varphi_1}{\lambda_1} + \varphi_2 + \varphi_3 + \frac{\varphi_4}{\lambda_4} - \varphi_0\left(2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_4}\right) + \frac{\varphi_2 - \varphi_4}{n(1+\lambda_4)} \tag{8}$$

For some problems it is preferable to use a reverse method in which the dependent and independent variables are interchanged. In two-dimensional potential flow, if φ and Ψ are considered to be the independent variables, x and y satisfy the Riemann-Cauchy equations

$$\frac{\partial x}{\partial \varphi} = \frac{\partial y}{\partial \psi} \qquad \frac{\partial x}{\partial \psi} = -\frac{\partial y}{\partial \varphi} \tag{9}$$

from which follow the inverse Laplace equations:

$$\frac{\partial^2 x}{\partial \varphi^2} + \frac{\partial^2 x}{\partial \psi^2} = 0 \qquad \frac{\partial^2 y}{\partial \varphi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0 \tag{10}$$

These equations can, of course, be expressed in terms of finite differences in the aforementioned manner.

Because the region of interest in fluid flow can always be bounded by a pair of potential lines (on which φ is constant) and a pair of stream lines (on which ψ is constant), no irregular stars will be encountered if the φ- ψ plane is chosen for computation. This advantage sometimes compensates for the drawback that the values of x and y along some boundary are unknown (though related) functions of φ. As will be illustrated later, the use of φ and ψ as independent variables is particularly advantageous if, as in free-jet problems, the boundary stream lines are not fixed a priori, but are determined by trial from a double boundary condition.

Relaxation Technique

The relaxation process can be described as a method of systematic refinement of an assumed variation of a function. In addition to the special techniques for the satisfying of various boundary conditions, two principal procedures are required: (a) the reduction of the residues, and (b) the subdivision of the network wherever the assumption of linear variation is found to be inaccurate.

The region of flow is first drawn to a large scale and then subdivided into squares and approximate values of the function selected for each intersection. For example, in the two-dimensional transition shown in Fig. 2, values of the stream function can be estimated from sketched stream lines once arbitrary values of ψ (0 and 90 in this instance) have been assigned for the center line and the boundary. The numerical values can be selected at will, because the results are to be presented dimensionlessly in terms of the flow characteristics either upstream or downstream from the transition.

Using Eqs. (2) and (7), one can compute from the assumed values of the function the value of the residue for each of the intersections. For example, near the point marked A in Fig. 2 the values could be as shown in Fig. 3, and the res-

idue as recorded to the right of the intersection of the lines. At A ,

$$a^2\nabla^2\psi = R = 37 + 49 + 33 + 19 - 4(36) = -6$$

The other values are similarly computed. In order to reduce

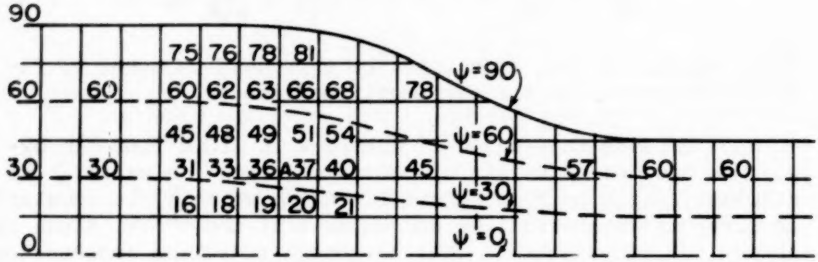

Fig. 2. Preliminary net for a two-dimensional transition

the residues it is necessary to adjust repeatedly the many values of ψ .

For a regular star, one for which all four legs are equal in length, a unit increase of ψ at a given point will result in a decrease of 4 in the corresponding residue as is apparent from Eq. (2), (5), or (6). Thus if ψ_A for Fig. 3 or 4, is decreased by 1, the residue is reduced from -6 to -2, as can be seen by referring to Eq. (2). In the same adjustment, the residues for each of the neighboring points are also decreased by 1 because ψ_A becomes ψ_4 in the computation of the residue at the point above. The process is therefore continued in accordance with the simple star patterns shown at the left of Fig. 4 for the several possible cases. Because each

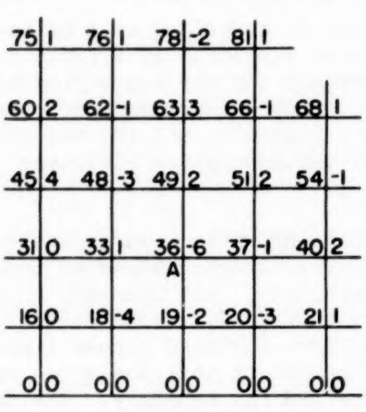

Fig. 3. Calculation of initial residues

change in ψ affects four other ψ-values directly and still others indirectly, the various values will need to be corrected many times. Large residues are first reduced without attempting to make them extremely small, refinement being sought only after the large discrepancies have been eliminated. In less than an hour of computation one can obtain a familiarity

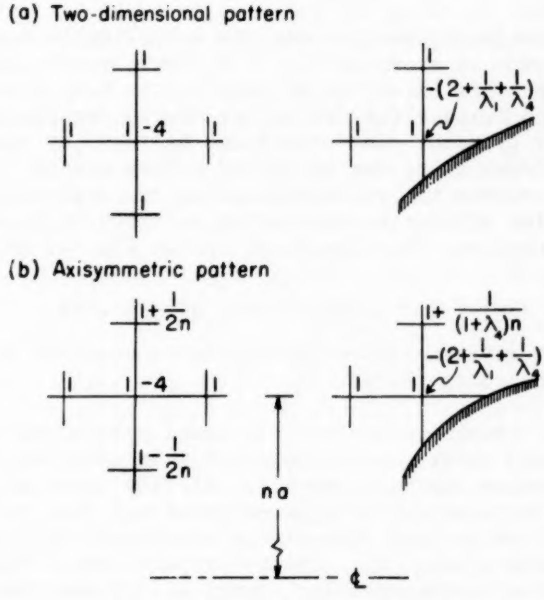

Fig. 4. Patterns for relaxation

with the simple process and a feeling for various short cuts such as over-correction and regional or block relaxation for

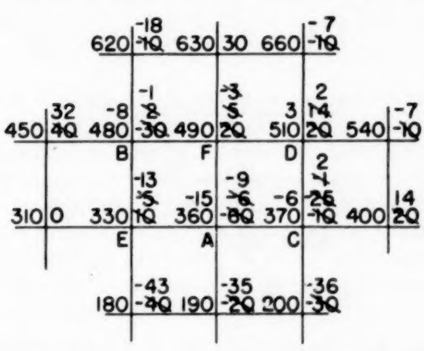

Fig. 5. Illustration of relaxation process (at points A-D)

which simple formulas can be derived. In Fig. 5 is shown a stage of calculation following single operations at each of points A-D (in alphabetical order). The original Ψ-values were obtained from Fig. 3, the values having been multiplied

by 10 to avoid the use of decimals.

Reduction of the residues to a negligible amount on a coarse net such as shown in Fig. 2 is usually not sufficient because errors are likely to be made in the many numerical processes and because the net is too coarse for the assumption of linearity to hold. The errors can be found by recomputing all the residues using the corrected values of the function, a difference between the recomputed value and that obtained from the successive alteration indicating an error in calculation or in bookkeeping. Thus for point A in Fig. 5, the calculation

$$364 + 490 + 330 + 190 - 4(345) = -6$$

based on the revised values constitutes a check on several of the preceding calculations.

Such errors once found can usually be eliminated rather rapidly and hence give no particular trouble unless residues are checked too infrequently. If they occur at a distance from the boundary in a two-dimensional flow pattern, corrections can be made directly in accordance with a predetermined pattern (Fig. 6). In the preparation of Fig. 6, it was assumed that a residue (or error) of 100 was found for the

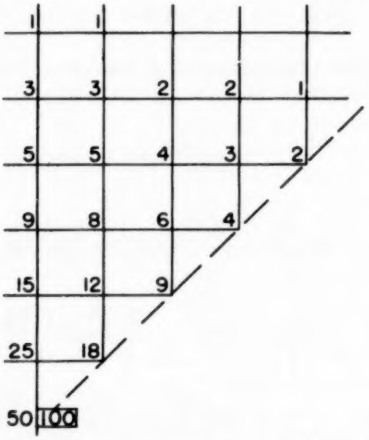

Fig. 6. Reduction of a single residue

point A and that those for all other points in the vicinity of A were negligibly small. In reducing R_A to zero, 128 of the neighboring values of the function were found to be affected by at least as much as 1% of R_A. For local residues other than 100, proportional values can be used, of course. The correction at any point is determined directly in terms of

the coordinates of the points; e.g., at a point three squares to the left and two squares below the point at which the error was made, the correction indicated in Fig. 6 would be 6% of the anomalous residue.

Because the initial grid is deliberately made coarse for simplicity, the variation of Ψ in some regions is probably not linear. The existence of this undesirable condition can be easily ascertained once the initial network has been satisfactorily completed. In a given region a value of Ψ can be obtained for the center of a square by averaging the four values at the corners (e.g., A, C, D, and F in Fig. 5). If this is done for each of four squares with a common vertex, the four new values can then be averaged to give a second value for the common point. If this value, say Ψ_A' , differs significantly from Ψ_A , a smaller spacing should be used. The necessary step, known as an advance to a finer net, need be taken only in the region for which non-linearity is evident from such computations. If successively finer nets are used only as required, the amount of work is kept to a minimum. Fortunately, the check for non-linearity and the first step in an advance are the same. As the work progresses, it is desirable to add one or more zeros to each of the numbers, thereby permitting further reduction of the residues without resorting to troublesome decimals. Once again, a comparatively small amount of experience is required to obtain an insight into the proper concepts for balancing the merits of additional refinements on a coarse net against those for an advance to a finer one. Naturally, the relaxation technique that has been described applies also if Φ and Ψ are used as independent variables.

The accuracy of a given result can only be judged by computing velocity or pressure distributions along a boundary or other arbitrarily selected line and determining whether they are both systematic and comparatively insensitive to further refinement. For this reason it is desirable to make such computations at various stages. It may be known that a small region of non-linearity remains, but if further advance does not alter appreciably the calculated distribution nearby, there is no need to continue.

<u>Boundary Conditions</u>

Little mention has been made of the method of satisfying a given set of boundary conditions because the procedure is actually a separate part of the problem, and because a variety of techniques is required. For the transition in Fig. 2, no difficulty exists. The values of Ψ along the solid bound-

ary and center line are fixed, and the net is simply, if laboriously, extended to the right and left until the condition of constant velocity ($\Psi = ky$) is satisfactorily approached. For other problems, however, special procedures are required. Among such conditions are those specifying (a) symmetry at the boundary, (b) free streamlines with or without gravitational effects, (c) the free surface in a seepage problem, and (d) the special condition for the inverse method. In some cases,

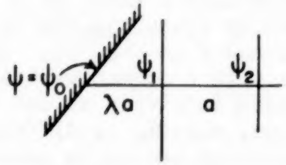

Fig. 7. Computation of x/ x at a boundary

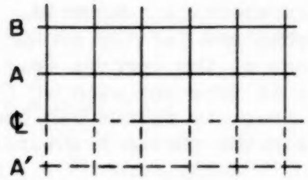

Fig. 8. Indication of reflection principle

the velocity at the boundary must be evaluated before the boundary condition can be assessed.

Along a solid boundary, the function φ is a variable and Ψ is a constant. If φ is used, the velocity can be computed directly for the midpoint of each interval along the boundary from the simple formula

$$v = -\frac{\partial \varphi}{\partial s} \approx -\frac{\varphi_1 - \varphi_4}{a\sqrt{\lambda_1^2 + \lambda_2^2}}$$

in which s represents distance along a streamline and φ_1 and φ_4 are adjacent values along the boundary as shown in Fig. 1b. If Ψ is used the normal gradient $\partial\Psi/\partial n$ must be calculated from $\partial\Psi/\partial x$ (or $\partial\Psi/\partial y$) which in turn is estimated from two values, one on the boundary and the other near the boundary. In such a case (see Fig. 7), the calculation

$$v_y = \frac{\partial \Psi}{\partial x} \approx \frac{\Psi_1 - \Psi_4}{\lambda_1 a}$$

yields an approximation of the y-component of the velocity at a point which is a distance $\lambda_1 a/2$ away from the boundary. Even though the assumption of local linearity approaches the actual conditions, better results are obtained by using a finite-difference formula for the derivative at the boundary based on three Ψ-values as illustrated in Fig. 7:

$$\frac{\partial \Psi}{\partial x} \approx \frac{1+\lambda}{\lambda}\,\frac{\Psi_1 - \Psi_0}{a} - \frac{\lambda}{1+\lambda}\,\frac{\Psi_2 - \Psi_0}{a} \qquad (11)$$

Inclusion of more than three terms was found to be unnecessary, the increased complexity of the computation not being accompanied by a significant increase in accuracy. Curvilinear extrapolation of this kind might seem unwarranted in a method which is based on the assumption of locally linear variation, but the resulting increased accuracy can obviate an advance to a finer net.

In the event that a flow pattern is symmetrical with respect to a center line, only one of the two halves need be determined, the line of symmetry being a streamline as in Fig. 3. If ψ is used, the corresponding boundary condition is automatically fixed as it is for a solid boundary, and residues which would be transferred to points on such a boundary are simply ignored. If, however, the velocity potential is used, the boundary values are not known but are subject to the condition that lines of constant φ must intersect the line of symmetry at right angles $(\partial\varphi/\partial n = 0)$. To satisfy this requirement it is only necessary to note that any residue coming to a point on the center line from a point on the neighboring line (designated as A in Fig. 8) would also come from the point A' symmetrically placed on the other side of the center, and should therefore be doubled. All other steps in the process remain unchanged, points on line B and on the center line being relaxed in the usual manner.

In the analysis of the patterns of flow occurring for various types of efflux, the free streamlines which are encountered necessitate special measures. Along a free streamline the pressure is constant; thus, for irrotational flow, the velocity is either constant, if gravitational effects are ignored, or proportional to the square root of the vertical distance below the line of total head, if gravitational effects are included. Initially the location of the free streamline is not known, and consequently must be assumed. The corresponding flow pattern is next determined through use of the relaxation process, and the velocities along the free streamlines are calculated. As the values obtained will not, in general, satisfy the specified boundary condition for the velocity, the assumed location of the free streamline must be changed and the process repeated. Because ψ is used for the direct method in this type of problem, the aforementioned method of evaluation $\partial\psi/\partial n$ by curvilinear extrapolation is extremely useful.

In the analysis of unconfined seepage through a porous medium, a somewhat different type of free streamline is encountered. As the piezometric head (velocity potential) is generally used, two conditions are required: (a) at every point along the free streamline the pressure must be zero, and

(b) the normal component of velocity must be zero. Accordingly, after a trial curve for the streamline bounding the flow has been drawn, values can be assigned for φ which automatically satisfy the condition that $p = 0$, that is, that φ is equal to the elevation. After the corresponding complete network has been obtained, the second boundary condition can be checked, and a revised bounding line drawn as required. Once

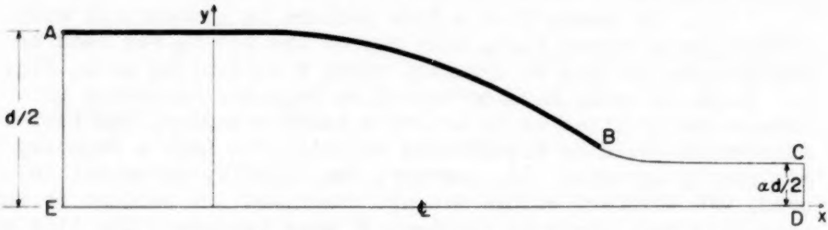

Fig. 9. Schematic diagram for reverse method

again, this process can be repeated, entirely or in part, until an acceptably correct result is obtained.

To check the imposed boundary condition or for evaluating the velocity along a curved or irregular boundary, $\partial\Psi/\partial n$ is evaluated from $\partial\Psi/\partial x$ or $\partial\Psi/\partial y$. With reference to Fig. 7, $\partial\Psi/\partial x$ is determined from Eq. (11) for the point on the bounding streamline. The value of $\partial\Psi/\partial n$ can then be determined from the expression

$$v = \frac{\partial\Psi}{\partial n} = \frac{1}{\sin\alpha}\frac{\partial\Psi}{\partial x} = \frac{1}{\cos\alpha}\frac{\partial\Psi}{\partial y}$$

in which α is the angle between the bounding streamline and the x-axis. For axisymmetric flow, the corresponding expression is

$$v = \frac{1}{r}\frac{\partial\Psi}{\partial n} = \frac{1}{r\sin\alpha}\frac{\partial\Psi}{\partial x} = \frac{1}{r\cos\alpha}\frac{\partial\Psi}{\partial y}$$

(No sign convention need be defined because the direction of the velocity is never in question.)

The treatment of the boundary condition if φ and Ψ are used as independent variables can best be explained by means of an example. On the surface of a jet issuing from a two-dimensional nozzle, the pressure and magnitude of the velocity are constants. This condition will determine the shape of the jet. With reference to Fig. 9, it is sufficient to consider the region ABCDE, where AE and CD are comparatively

far away from the contraction. Along CD, φ can be assumed to be a constant, say zero. The corresponding value for x can be taken from Fig. 9, and is denoted by X. Taking $\psi = 0$ along ED, one has $\psi = -q/2$ along ABC, q being the discharge per unit width of the nozzle. The value of φ along AE can be taken to be a sufficiently large constant K that the flow downstream from the contraction is essentially uniform, but it should be remembered that so far the corresponding value of x (denoted by X') has not been determined. Aside from the somewhat complicated boundary conditions for x and y along ABC, the conditions at the boundaries are:

(1) for $\varphi = 0$: $x = X$ (known), $y = -\psi d/q$
(2) for $\varphi = K$: $x = X'$(unknown), $y \sim \psi$
(3) for $\psi = 0$: $\partial x/\partial\psi = 0$, $y = 0$

in which d is the width of the nozzle at A.

Along AB it would be possible to translate the condition

$$\frac{\partial\varphi}{\partial n} = 0$$

(n being measured in a direction normal to AB) into one involving x or y as the unknowns, but the result is extremely unwieldy. Instead, one demands the satisfaction of one of the Cauchy-Riemann equations (Eq. (9)):

$$\frac{\partial x}{\partial\varphi} = \frac{\partial y}{\partial\psi}$$

and requires that

$$x = x(\varphi) \quad \text{and} \quad y = y(\varphi) \tag{12}$$

be equivalent to

$$y = f(x) \tag{13}$$

which describes the boundary AB.

The conditions along BC can be similarly formulated. One again demands the satisfaction of one of the relationships from Eq. (9) along $\psi = -q/2$. But instead of Eqs. (12), one requires

$$\left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 = \left(\frac{\alpha d}{q}\right)^2 \tag{14}$$

in which α is the ratio of the ultimate width of the jet to d.

With reference to Fig. 10, the procedure is then as follows. Assuming $x(\varphi)$ along AC, one obtains X' and can locate B where x is known. From $y = f(x)$, the value of y along AB can be obtained. The value of y from B to C is obtained by integrating Eq. (14) numerically, α being assumed such that the calculated y at C will be equal to

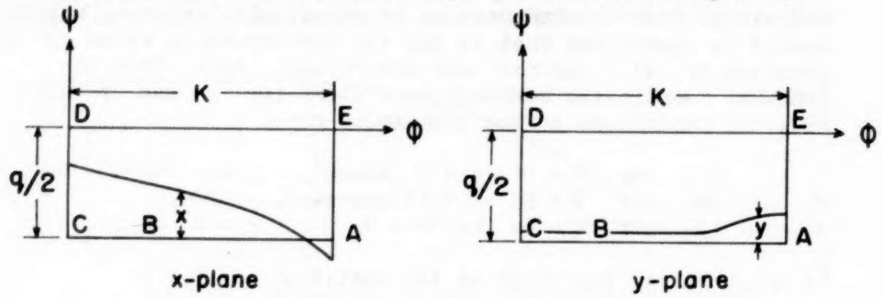

Fig. 10. Work sheets for the evaluation of x and y

$\alpha d/2$. With the value of y at C given, the boundary condition for y at $\varphi = 0$ is determined from the linear relation between y and ψ along CD.

After the relaxation procedure for y is completed, the result is checked against one of Eqs. (9). If this equation is not satisfied, $x(\varphi)$ is corrected accordingly. The process is repeated until it is satisfied. Finally, x can be computed either by one relaxation process, or directly by using Eqs. (9).

In the assumption of $x(\varphi)$ along AC, it should be remembered that $x(\varphi)$ is asymptotically linear. At $\varphi = 0$, $\partial x/\partial\varphi$ is approximately $-\alpha d/q$ and y is approximately $d/2$. At $\varphi = K$, $\partial x/\partial\varphi$ is approximately d/q , and y is approximately equal to $d/2$. The final result of a computation for the two-dimensional nozzle with the boundary geometry given in Fig. 9 is presented in the following section.

Once the various steps in the relaxation process, as outlined above, have been completed, any required results can be obtained: the shape of a free streamline is established in the process, the pressure and velocity distributions can be determined throughout space with reference to the pressure and velocity in a region of uniform flow, comparative quantities of seepage can be evaluated, and coefficients of discharge can be computed. The one remaining question is the degree of accuracy obtained - a difficult one to answer.

There is no direct way to assess the accuracy other than to make determinations of, for example, velocity distributions from each of the various nets prepared. If the change in this result following a particular refinement is comparatively small one can conclude that the result is satisfactory.

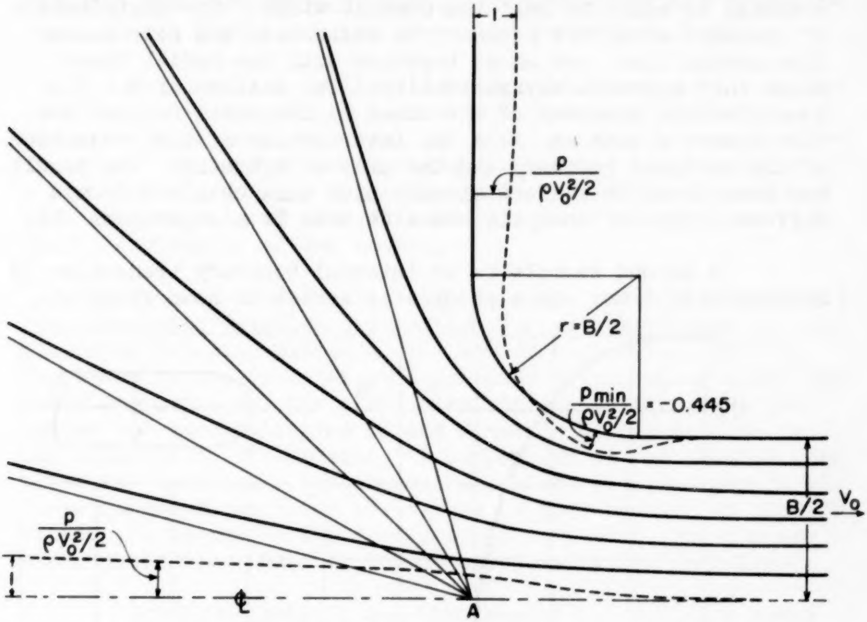

Fig. 11. Two-dimensional curved inlet (S. Ince)

Various ways of resolving partially this troublesome matter will become evident in each problem. It is a significant fact that no two regions are entirely equivalent. In one region several subdivisions may be required, in another none. Furthermore, even sizeable errors in the pattern at one point may have a very small effect at another. As the entire process is time consuming, special care should be taken to assess at each stage of the calculation the reliability of the results.

Applications

The importance of the relaxation method is best indicated by its diversity of application which is evident from the following series of examples for flow through boundary transitions, flow with a free surface, and seepage. The patterns are two-dimensional or axisymmetric. As the details of the calculation procedure for the interior are similar in all problems, only in the satisfying of the boundary conditions do

the problems differ significantly.

The results of a calculation of the flow through a two-dimensional inlet transition between a reservoir and a conduit are shown in Fig. 11; in this instance the radius of rounding is equal to half the conduit width. The distribution of pressure along the boundary is indicated, and representative streamlines are shown together with the radial lines which they approach asymptotically. The pattern of the flow some distance upstream of the inlet is identical to that for flow toward a sink at A - the intersection of the projection of the vertical boundary and the axis of symmetry. The result has been found to conform closely with that obtained from a different type of analysis and with that from experiment [2].

A second example is an internal boundary transition in axisymmetric flow. In a study of a series of head forms [3],

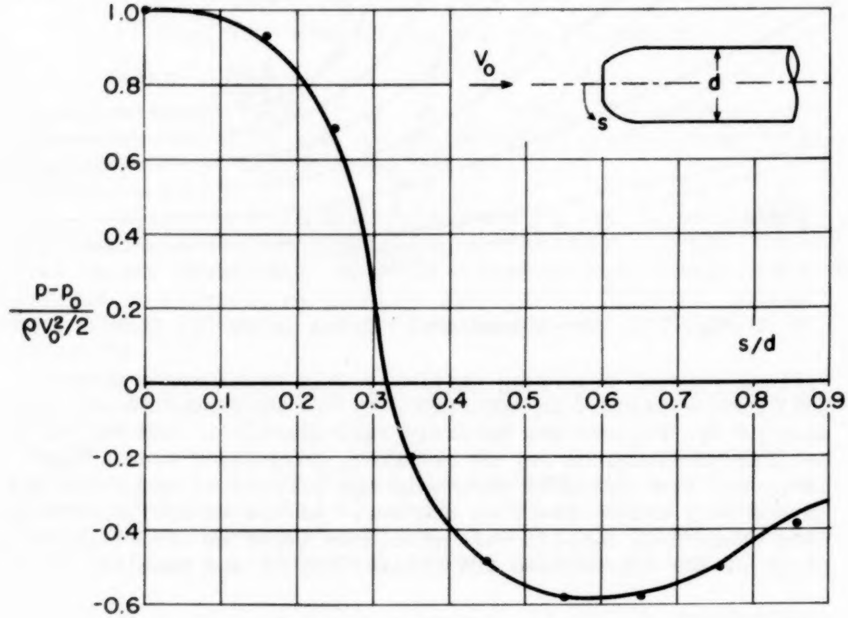

Fig. 12. Pressure distribution for axisymmetric flow past modified ellipsoidal head form (W. Hsieh)

the pressure distribution was determined for a modified ellipsoidal form as indicated in Fig. 12. A comparison is made of the results obtained from the Institute water tunnel with those calculated numerically. The correspondence is good from the point of stagnation to a point somewhat beyond the region

of minimum pressure. The dimensionless value for the minimum pressure is -0.57, which can be compared with a value of -0.74 for a hemispherical nose form. The significance of this comparison is that a body with the modified nose form could move at higher velocities without producing cavitation than could one with the hemispherical form. The effects of the boundary-layer growth in this region are evidently very small.

Applications of the relaxation technique to the determination of flow patterns involving a free surface include problems of efflux, overflow, jet impingement, and steady cavity flow. Each of these is characterized by the existence of a boundary along which the pressure is constant. These have been the subjects of several calculations which serve as further illustration of the method.

From extensive calculations, Rouse and Abul-Fetouh [4] have presented detailed information for efflux through an orifice symmetrically placed at the end of a circular conduit. The shape of the free jet, the pressure distribution along the solid boundary, and the coefficient of contraction were presented for various ratios of the diameter of the orifice to the diameter of the conduit. A comparison of the results with those for comparable two-dimensional flows revealed remarkably close correspondence for the values of the contraction coefficient. Furthermore, the coefficients available from various experiments were found to follow essentially the same trend.

Overflow problems are inaccessible to ordinary methods of hydrodynamics because of the paramount importance of gravity. That is, surfaces of constant pressure are not, for these flows, also surfaces of constant velocity. Thus even the classical method of the free-streamline theory is not applicable. Southwell [1] has presented a calculation for the free overfall. Also, Citrini [5] recently published the results of a calculation for a circular weir which are applicable to the design of a morning-glory spillway; he obtained the pattern for the case in which the (horizontal) radius of curvature of the weir is ten times the total head on the weir.

The results of a refined calculation for two-dimensional flow over a very high weir, conducted at the Iowa Institute of Hydraulic Research, are shown in Fig. 13. Once again, the flow a great distance to the left could be taken as that resulting from a sink at the intersection of the plane of the weir with the surface of total head. Far to the right the streamlines approach a trajectory of free fall, and the internal pressure approaches zero. The significant region - near the crest of the weir - is shown at three different scales in the various parts of the figure. The coefficient of discharge

C_d in the conventional equation

$$Q = C_d \sqrt{2g}\ L\ H^{3/2} \tag{15}$$

was found to be 0.408. This value is somewhat more, as it should be, than the value of 0.388 found by Citrini for the curved weir. Both free surfaces and representative internal streamlines are shown. As the shape of the former is signifi-

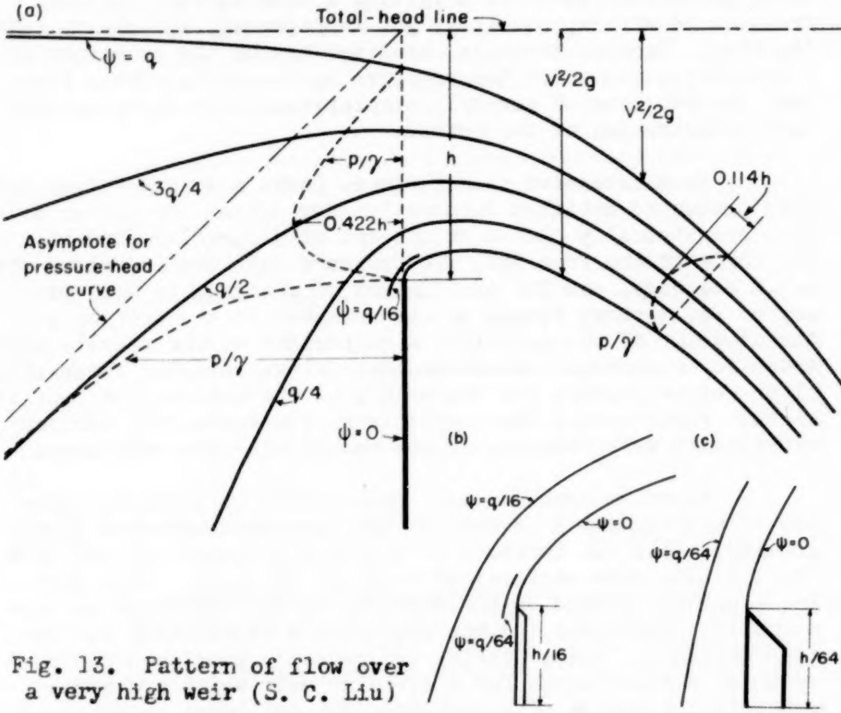

Fig. 13. Pattern of flow over a very high weir (S. C. Liu)

cant in the design of spillway, coordinates of the two bounding curves are given in the accompanying table in their ratio to the head.

Perhaps the longest of all the calculations described herein, performed by the second author of this paper as a part of his doctoral dissertation, was that for axisymmetric flow with an internal boundary which was partly fixed and partly a steady-state cavity. Once again, this calculation duplicated part of the experiments described in reference [3], being for flow around a hemispherical head form mounted on a shaft of equal diameter. The cavitation index

$$\sigma = \frac{p_0 - p_v}{\rho V_0^2/2}$$

for this case was 0.2, a sufficiently low value that a large vapor cavity formed. The pressure distribution observed in the experiments was imposed as an internal boundary condition,

Coordinates of Free Surfaces for Flow over a High Weir

8x/H	y/H Upper Nappe	8x/H	y/H Upper Nappe	y/H Lower Nappe
-16	0.991	0	0.855	0
-15	0.989	1/32	. . .	0.013
-14	0.988	1/16	. . .	0.020
-13	0.985	1/8	. . .	0.031
-12	0.983	1/4	. . .	0.047
-11	0.979	1/2	. . .	0.068
-10	0.973	1	0.822	0.091
- 9	0.969	2	0.781	0.113
- 8	0.965	3	0.733	0.101
- 7	0.959	4	0.678	0.065
- 6	0.951	5	0.609	0.010
- 5	0.943	6	0.531	-0.058
- 4	0.932	7	0.422	-0.135
- 3	0.921	8	0.340	-0.222
- 2	0.903	9	0.230	-0.323
- 1	0.880	10	0.114	-0.432
		11	-0.011	-0.561
		12	-0.149	-0.705
		13	-0.305	-0.866
		14	-0.491	-1.043
		15	-0.694	
		16	-0.910	

and the corresponding shape of the vapor cavity was determined by successive approximation.

In Fig. 14 are shown (a) the observed and computed cavity profiles, (b) the pressure distribution, and (c) the pattern used for the calculation. Further details of the calculation are available in thesis form [6], including the Ψ - values at each of the intersections in Fig. 14c. The conditions far upstream or far downstream were, of course, those for parallel flow. By means of an artifice, the radial extent of the net was restricted to ten times the radius of the body.

axis. This was the topic of an investigation at the Iowa Institute by A. LeClerc [7]. In this study, the form of the free surface was determined by electrical analogy, only the internal flow pattern being evaluated by relaxation. The velocity was assumed to be sufficiently high that the effect of gravity as well as those of viscosity could be neglected. The shape of the jet, representative streamlines, the variation of the pressure along the axis and along the plate, and the limiting curve for the free surface are shown in Fig. 15.

A characteristic application of the inverse method is shown in Fig. 16. The pattern of efflux from a curved two-

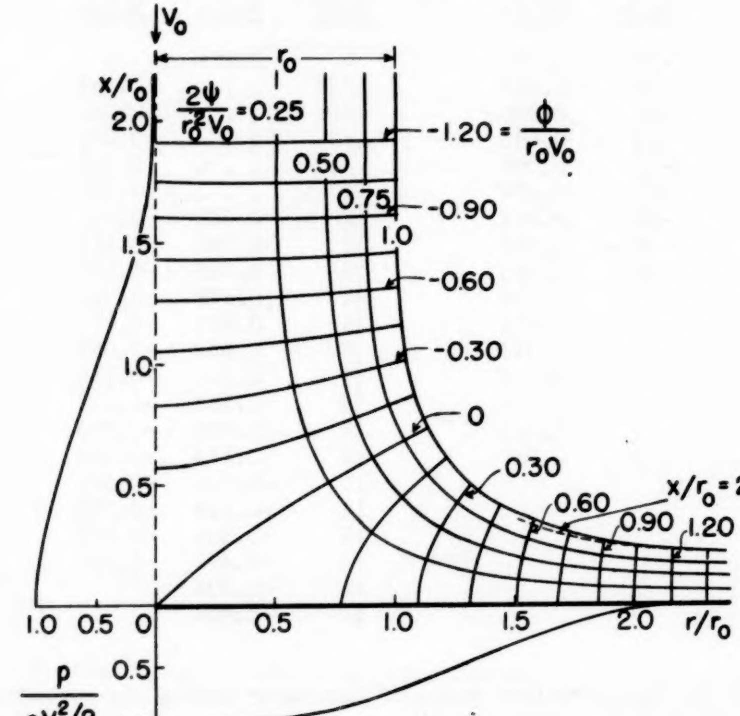

Fig. 15. Impingement of a jet on a flat plate (A. LeClerc)

dimensional nozzle is that which was also used in explaining the techniques. An arbitrary parabolic curve at the end of a parallel approach reduces the normal section at the region of efflux to one third of that of the approach. The contraction coefficient was found to be 0.75. A point which is worth re-

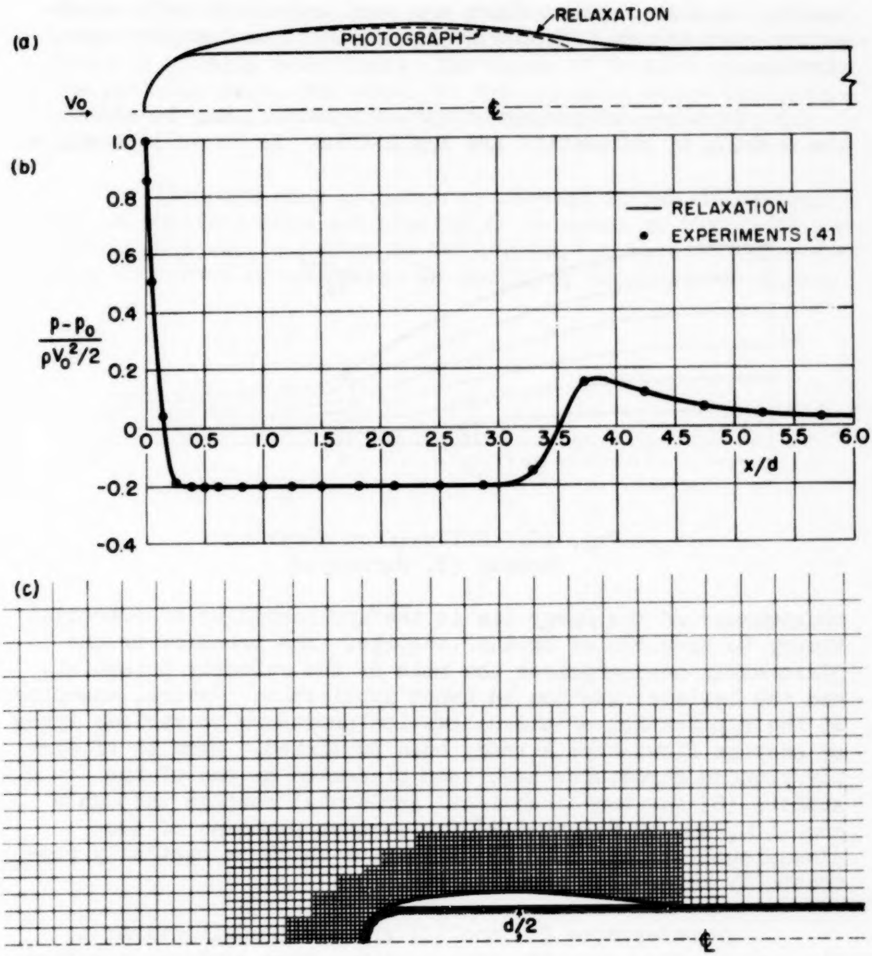

Fig. 14. Flow with cavitation for a hemispherical head form (a) cavitation pocket, (b) pressure distribution, and (c) relaxation pattern (E. Y. Hsu)

The values of Ψ some distance (radially) away from the body were computed for an axisymmetric half body of equivalent size and location. It was then necessary to extend the net only until its values coincided with those calculated for the half body. The resultant saving in time was a considerable one.

A third type of free-surface flow is typified by the impingement of a circular jet on a plate normal to the jet

peating is that regular stars are used throughout this calculation even though both the solid and the free boundary are curved.

Problems of seepage comprise a third group to which the methods of relaxation are applicable. As is well known, a

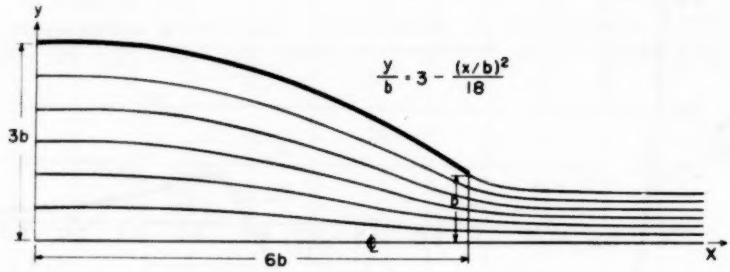

Fig. 16. Efflux from a curved nozzle (T. Sarpkaya)

consequence of the Darcy law is the applicability of potential theory to problems of laminar seepage. The pressure or the piezometric head acquires the role of the velocity potential, and the Laplace equation is hence applicable. Several examples of the application of the relaxation procedure to various types of seepage flows have already been published. Results of simply bounded structures have been presented by one of the authors [8] for two-dimensional horizontal seepage through a monolith, and by McNamee [9] for various patterns of two-dimensional seepage into an excavation which is partially protected by impervious sheet piling.

In references [1] and [8] are given applications to the more complex case of seepage with a free surface, a refined version of the latter being presented in Fig. 17. These are cases for which, once again, the bounding streamline is initially unknown, and can be treated only by trial and error. The two boundary conditions for the free surface are: (a) the direction of the resultant velocity must coincide with that of the free streamline, and (b) the free surface must be one of constant pressure (i.e., the piezometric head, which is φ , is equal to the elevation). The second of these was assumed for a trial curve and a relaxation of the interior was completed. Adjustment of the free surface was then made on the basis of the shortcomings with regard to the first condition. As the φ-function was used throughout, the boundary condition for

the horizontal bottom line was that of orthogonality (or symmetry), and was fulfilled by the application of double residues as already described. The value of φ is a constant on the upstream face, but equal to the distance below the intersection of the free surface and the downstream face on the latter.

Although many problems of seepage in two dimensions can be obtained more satisfactorily by means of classical methods, only such a method as relaxation can be used for complex or curved boundaries. In addition, axisymmetric flows,

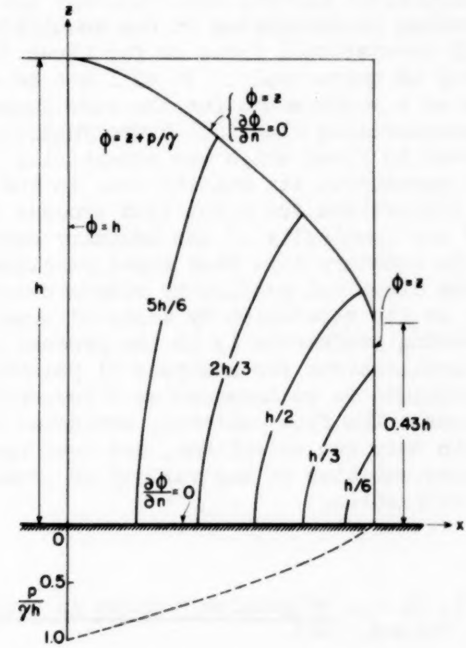

Fig. 17. Seepage through a vertical wall (C. S. Yih)

such as flow of ground water to a well or the axisymmetric counterpart of McNamee's problem of seepage into an excavation, are beyond the scope of ordinary methods.

Résumé

In the presentation both of the relaxation method and of examples in which it is utilized, the utility and flexibility of the process have been amply demonstrated. Fortunately,

the technique can be quickly mastered, containing as it does only two basic parts. The mechanism of the internal reduction of residues is simple and extremely repetitive. The various boundary conditions are either assigned initially or met indirectly as a part of the process. An advantage of the method is that absolute accuracy is not a requirement; an occasional error once made will be caught and need not introduce an excessive loss of time. Although the method is simple, it is not completely mechanical, so that the exercise of judgment and the introduction of suitable modifications of a given routine can greatly reduce the time required.

The method is restricted to the resolution of problems of essentially irrotational flows or for flows for which the Reynolds number is quite small. It will not be and should not be considered as a replacement for the more general analytical solutions characterizing classical hydrodynamics. Although still restricted to flows which are essentially two dimensional or axially symmetric, its utility lies in the fact that within these limitations the relaxation process can be used regardless of the complexity of the boundary condition. A small change in boundary form that might preclude the solving of an otherwise classical problem by standard methods presents no difficulty in its resolution by means of numerical methods. As the engineering profession is in the process of discovering many new applications for analyses of potential flows, the relaxation technique is recommended as a powerful adjunct to standard methods. The flow patterns, presented herein for the first time with only two exceptions, are both useful in themselves and representative of the variety of problems which can be solved by relaxation.

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